Résolution de systèmes : application à la détermination de sous-espaces propres

Exercice 1

On considère les matrices suivantes.

a) $M_1 = \begin{pmatrix} 2 & -2 & 0 \\ 0 & 6 & -6 \\ 0 & 0 & 12 \end{pmatrix}$

b) $M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

c) $M_3 = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 2 \\ 1 & -1 & 3 \end{pmatrix}$

d) $M_4 = \begin{pmatrix} 5 & 1 & -1 \\ 2 & 4 & -2 \\ 1 & -1 & 3 \end{pmatrix}$

e) $M_5 = \begin{pmatrix} 3 & -1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

f) $M_6 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Pour tout $i \in [\![1, 6]\!]$ et tout $\lambda \in \mathbb{R}$, on note :

$$E_\lambda(M_i) = \{X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R}) \mid (M_i - \lambda I)\ X = 0\}$$

a) (i) Déterminer $E_2(M_1)$.

(ii) Déterminer $E_6(M_1)$.

(iii) Déterminer $E_{12}(M_1)$.

b) Déterminer $E_1(M_2)$.

c) Déterminer $E_2(M_3)$.

d) (i) Déterminer $E_2(M_4)$.

(ii) Déterminer $E_4(M_4)$.

(iii) Déterminer $E_6(M_4)$.

e) Déterminer $E_2(M_5)$.

f) (i) Déterminer $E_0(M_6)$.

(ii) Déterminer $E_3(M_6)$.

Démonstration.

a) (i) Soit $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R})$.

$$\begin{array}{rcl} X \in E_2(M_1) & \Longleftrightarrow & M_1 X = 2X \\ & \Longleftrightarrow & (M_1 - 2I)\ X = 0_{\mathscr{M}_{3,1}(\mathbb{R})} \\ & \Longleftrightarrow & \begin{pmatrix} 0 & -2 & 0 \\ 0 & 4 & -6 \\ 0 & 0 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ & \Longleftrightarrow & \left\{ \begin{array}{rcrcl} -2y & & & = & 0 \\ 4y & - & 6z & = & 0 \\ & & 10z & = & 0 \end{array} \right. \\ & \underset{L_2 \leftarrow L_2 + 2L_1}{\Longleftrightarrow} & \left\{ \begin{array}{rrcl} -2\,y & & = & 0 \\ & -6\,z & = & 0 \\ & 10\,z & = & 0 \end{array} \right. \\ & \underset{L_3 \leftarrow 3L_3 + 5L_2}{\Longleftrightarrow} & \left\{ \begin{array}{rrcl} -2\,y & & = & 0 \\ & -6\,z & = & 0 \\ & 0 & = & 0 \end{array} \right. \end{array}$$

On en déduit :

$$\begin{array}{rcl} E_2(M_1) & = & \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R}) \mid y = 0 \ \text{ et } \ z = 0 \right\} \\ & = & \left\{ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \mid x \in \mathbb{R} \right\} = \left\{ x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mid x \in \mathbb{R} \right\} = \operatorname{Vect}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \end{array}$$

$$\boxed{E_2(M_1) = \operatorname{Vect}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right)}$$

(ii) Soit $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R})$.

$$\begin{array}{rcl} X \in E_6(M_1) & \Longleftrightarrow & M_1 X = 6X \\ & \Longleftrightarrow & (M_1 - 6I)\ X = 0_{\mathscr{M}_{3,1}(\mathbb{R})} \\ & \Longleftrightarrow & \begin{pmatrix} -4 & -2 & 0 \\ 0 & 0 & -6 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ & \Longleftrightarrow & \left\{ \begin{array}{rcrcrcl} -4x & - & 2y & & & = & 0 \\ & & & - & 6z & = & 0 \\ & & & & 6z & = & 0 \end{array} \right. \\ & \underset{L_3 \leftarrow L_3 + L_2}{\Longleftrightarrow} & \left\{ \begin{array}{rcrcrcl} -4x & - & 2y & & & = & 0 \\ & & & - & 6z & = & 0 \\ & & & & 0 & = & 0 \end{array} \right. \\ & \Longleftrightarrow & \left\{ \begin{array}{rcrcl} -4x & & & = & 2y \\ & - & 6z & = & 0 \end{array} \right. \end{array}$$

On en déduit :

$$\begin{array}{rcl} E_6(M_1) & = & \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R}) \mid x = -\frac{1}{2}y \ \text{ et } \ z = 0 \right\} \\ & = & \left\{ \begin{pmatrix} -\frac{1}{2}y \\ y \\ 0 \end{pmatrix} \mid y \in \mathbb{R} \right\} \\ & = & \left\{ x \cdot \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \mid y \in \mathbb{R} \right\} = \operatorname{Vect}\left(\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \right) \end{array}$$

$$\boxed{E_6(M_1) = \operatorname{Vect}\left(\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right)}$$

(iii) Soit $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R})$.

$$\begin{array}{rcl}
X \in E_{12}(M_1) & \iff & M_1 X = 12X \\
& \iff & (M_1 - 12I)\, X = 0_{\mathscr{M}_{3,1}(\mathbb{R})} \\
& \iff & \begin{pmatrix} -10 & -2 & 0 \\ 0 & -6 & -6 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
& \iff & \left\{ \begin{array}{rcrcrcl} -10x & - & 2y & & & = & 0 \\ & - & 6y & - & 6z & = & 0 \\ & & & & 0 & = & 0 \end{array} \right. \\
& \iff & \left\{ \begin{array}{rcrcr} -10x & - & 2y & = & 0 \\ & - & 6y & = & 6z \end{array} \right. \\
& \underset{}{\overset{L_1 \leftarrow 3\,L_1 - L_2}{\iff}} & \left\{ \begin{array}{rcrcr} -30x & & & = & -6z \\ & - & 6y & = & 6z \end{array} \right.
\end{array}$$

On en déduit :

$$\begin{array}{rcl}
E_{12}(M_1) & = & \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R}) \mid x = \frac{1}{5} z \ \text{ et } \ y = -z \right\} \\
& = & \left\{ \begin{pmatrix} \frac{1}{5} z \\ -z \\ z \end{pmatrix} \mid z \in \mathbb{R} \right\} \\
& = & \left\{ z \cdot \begin{pmatrix} \frac{1}{5} \\ -1 \\ 1 \end{pmatrix} \mid z \in \mathbb{R} \right\} \ = \ \mathrm{Vect}\left(\begin{pmatrix} \frac{1}{5} \\ -1 \\ 1 \end{pmatrix} \right)
\end{array}$$

$$\boxed{E_{12}(M_1) = \mathrm{Vect}\left(\begin{pmatrix} 1 \\ -5 \\ 5 \end{pmatrix} \right)}$$

b) Soit $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R})$.

$$\begin{array}{rcl}
X \in E_1(M_2) & \iff & M_2 X = X \\
& \iff & (M_2 - I)\, X = 0_{\mathscr{M}_{3,1}(\mathbb{R})} \\
& \iff & \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
& \iff & \left\{ \begin{array}{rcrcl} & & 0 & = & 0 \\ x & + & z & = & 0 \\ & & 0 & = & 0 \end{array} \right. \\
& \iff & \left\{ \ x \ = \ -z \right.
\end{array}$$

On en déduit :

$$\begin{array}{rcl}
E_1(M_2) & = & \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R}) \mid x = -z \right\} \\
& = & \left\{ \begin{pmatrix} -z \\ y \\ z \end{pmatrix} \mid (y, z) \in \mathbb{R}^2 \right\} \\
& = & \left\{ y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \mid (y, z) \in \mathbb{R}^2 \right\} \\
& = & \mathrm{Vect}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)
\end{array}$$

$$\boxed{E_1(M_2) = \mathrm{Vect}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)}$$

c) Déterminons $E_2(M_3)$. Soit $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R})$.

$$\begin{array}{rcl}
X \in E_2(M_3) & \Longleftrightarrow & M_3 X = 2\,X \\
& \Longleftrightarrow & (M_3 - 2I)\,X = 0_{\mathscr{M}_{3,1}(\mathbb{R})} \\
& \Longleftrightarrow & \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
& \Longleftrightarrow & \left\{ \begin{array}{rcrcrcl} x & - & y & + & z & = & 0 \\ 2\,x & - & 2\,y & + & 2\,z & = & 0 \\ x & - & y & + & z & = & 0 \end{array} \right. \\
& \underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - 2\,L_1}{\Longleftrightarrow}} & \left\{ \begin{array}{rcrcrcl} x & - & y & + & z & = & 0 \\ & & & & 0 & = & 0 \\ & & & & 0 & = & 0 \end{array} \right. \\
& \Longleftrightarrow & \{\ x = y - z
\end{array}$$

On en déduit :

$$\begin{array}{rcl}
E_2(M_3) & = & \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R}) \mid x = y - z \right\} \\
& = & \left\{ \begin{pmatrix} y - z \\ y \\ z \end{pmatrix} \mid (y,z) \in \mathbb{R}^2 \right\} \\
& = & \left\{ y \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \mid (y,z) \in \mathbb{R}^2 \right\} \ = \ \mathrm{Vect}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)
\end{array}$$

$$\boxed{E_2(M_3) = \mathrm{Vect}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)}$$

d) (i) Déterminons $E_2(M_4)$. Soit $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R})$.

$$\begin{array}{rcl}
X \in E_2(M_4) & \Longleftrightarrow & M_4 X = 2\,X \\
& \Longleftrightarrow & (M_4 - 2I)\,X = 0_{\mathscr{M}_{3,1}(\mathbb{R})} \\
& \Longleftrightarrow & \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
& \Longleftrightarrow & \left\{ \begin{array}{rcrcrcl} 3\,x & + & y & - & z & = & 0 \\ 2\,x & + & 2\,y & - & 2\,z & = & 0 \\ x & - & y & + & z & = & 0 \end{array} \right. \\
& \underset{L_3 \leftarrow 3\,L_3 - L_1}{\overset{L_2 \leftarrow 3\,L_2 - 2\,L_1}{\Longleftrightarrow}} & \left\{ \begin{array}{rcrcrcl} 3\,x & + & y & - & z & = & 0 \\ & & 4\,y & - & 4\,z & = & 0 \\ & - & 4\,y & + & 4\,z & = & 0 \end{array} \right. \\
& \overset{L_3 \leftarrow L_3 + L_2}{\Longleftrightarrow} & \left\{ \begin{array}{rcrcrcl} 3\,x & + & y & - & z & = & 0 \\ & & 4\,y & - & 4\,z & = & 0 \\ & & & & 0 & = & 0 \end{array} \right. \\
& \Longleftrightarrow & \left\{ \begin{array}{rcrcl} 3\,x & + & y & = & z \\ & & 4\,y & = & 4\,z \end{array} \right. \\
& \overset{L_1 \leftarrow 4\,L_1 - L_2}{\Longleftrightarrow} & \left\{ \begin{array}{rcl} 12\,x & = & 0 \\ 4\,y & = & 4\,z \end{array} \right.
\end{array}$$

On en déduit :

$$\begin{array}{rcl}
E_2(M_4) & = & \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R}) \mid x = 0 \ \text{ et } \ y = z \right\} \\
& = & \left\{ \begin{pmatrix} 0 \\ z \\ z \end{pmatrix} \mid z \in \mathbb{R} \right\} \ = \ \left\{ z \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \mid z \in \mathbb{R} \right\} \ = \ \mathrm{Vect}\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right)
\end{array}$$

(ii) Déterminons $E_4(M_4)$. Soit $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R})$.

$$\begin{aligned}
X \in E_4(M_4) &\iff M_4 X = 4\,X \\
&\iff (M_4 - 4I)\,X = 0_{\mathscr{M}_{3,1}(\mathbb{R})} \\
&\iff \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&\iff \left\{ \begin{array}{rcrcrcl} x & + & y & - & z & = & 0 \\ 2\,x & & & - & 2\,z & = & 0 \\ x & - & y & - & z & = & 0 \end{array} \right. \\
&\underset{\substack{L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - L_1}}{\iff} \left\{ \begin{array}{rcrcrcl} x & + & y & - & z & = & 0 \\ & - & 2\,y & & & = & 0 \\ & - & 2\,y & & & = & 0 \end{array} \right. \\
&\underset{L_3 \leftarrow L_3 - L_2}{\iff} \left\{ \begin{array}{rcrcrcl} x & + & y & - & z & = & 0 \\ & - & 2\,y & & & = & 0 \\ & & & & 0 & = & 0 \end{array} \right. \\
&\iff \left\{ \begin{array}{rcrcl} x & + & y & = & z \\ & - & 2\,y & = & 0 \end{array} \right. \\
&\underset{L_1 \leftarrow 2\,L_1 + L_2}{\iff} \left\{ \begin{array}{rcrcl} 2\,x & & & = & 2\,z \\ & & y & = & 0 \end{array} \right.
\end{aligned}$$

On en déduit :

$$\begin{aligned}
E_4(M_4) &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R}) \mid x = z \ \text{ et } \ y = 0 \right\} \\
&= \left\{ \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} \mid z \in \mathbb{R} \right\} = \left\{ z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \mid z \in \mathbb{R} \right\} = \operatorname{Vect}\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right)
\end{aligned}$$

(iii) Déterminons $E_6(M_4)$. Soit $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R})$.

$$\begin{aligned}
X \in E_6(M_4) &\iff M_4 X = 6\,X \\
&\iff (M_4 - 6I)\,X = 0_{\mathscr{M}_{3,1}(\mathbb{R})} \\
&\iff \begin{pmatrix} -1 & 1 & -1 \\ 2 & -2 & -2 \\ 1 & -1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&\iff \left\{ \begin{array}{rcrcrcl} -x & + & y & - & z & = & 0 \\ 2\,x & - & 2\,y & - & 2\,z & = & 0 \\ x & - & y & - & 3\,z & = & 0 \end{array} \right. \\
&\underset{\substack{L_2 \leftarrow L_2 + 2L_1 \\ L_3 \leftarrow L_3 + L_1}}{\iff} \left\{ \begin{array}{rcrcrcl} -x & + & y & - & z & = & 0 \\ & & & - & 4\,z & = & 0 \\ & & & - & 4\,z & = & 0 \end{array} \right. \\
&\underset{L_3 \leftarrow L_3 - L_2}{\iff} \left\{ \begin{array}{rcrcrcl} -x & + & y & - & z & = & 0 \\ & & & - & 4\,z & = & 0 \\ & & & & 0 & = & 0 \end{array} \right. \\
&\iff \left\{ \begin{array}{rcrcl} -x & - & z & = & -y \\ & - & 4\,z & = & 0 \end{array} \right. \\
&\underset{L_1 \leftarrow 4\,L_1 - L_2}{\iff} \left\{ \begin{array}{rcrcl} -4\,x & & & = & -4\,y \\ & - & 4\,z & = & 0 \end{array} \right.
\end{aligned}$$

On en déduit :

$$\begin{aligned}
E_6(M_4) &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R}) \mid x = y \ \text{ et } \ z = 0 \right\} \\
&= \left\{ \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} \mid y \in \mathbb{R} \right\} = \left\{ y \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \mid y \in \mathbb{R} \right\} = \operatorname{Vect}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right)
\end{aligned}$$

e) Déterminons $E_2(M_5)$. Soit $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R})$.

$$\begin{array}{rcl}
X \in E_2(M_5) & \iff & M_5X = 2\,X \\
& \iff & (M_5 - 2I)\,X = 0_{\mathscr{M}_{3,1}(\mathbb{R})} \\
& \iff & \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
& \iff & \left\{ \begin{array}{rcrcrcl} x & - & y & + & z & = & 0 \\ x & & & & & = & 0 \\ & & y & - & z & = & 0 \end{array} \right. \\
& \underset{L_2 \leftarrow L_2 - L_1}{\iff} & \left\{ \begin{array}{rcrcrcl} x & - & y & + & z & = & 0 \\ & & y & - & z & = & 0 \\ & & y & - & z & = & 0 \end{array} \right. \\
& \underset{L_3 \leftarrow L_3 - L_2}{\iff} & \left\{ \begin{array}{rcrcrcl} x & - & y & + & z & = & 0 \\ & & y & - & z & = & 0 \\ & & & & 0 & = & 0 \end{array} \right. \\
& \iff & \left\{ \begin{array}{rcrcr} x & - & y & = & -z \\ & & y & = & z \end{array} \right. \\
& \underset{L_1 \leftarrow L_1 + L_2}{\iff} & \left\{ \begin{array}{rcrcl} x & & & = & 0 \\ & & y & = & z \end{array} \right.
\end{array}$$

On en déduit :

$$\begin{array}{rcl}
E_2(M_5) & = & \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid M_5X = 2X \right\} \\
& = & \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x = 0 \ \text{ et } \ y = z \right\} \ = \ \left\{ \begin{pmatrix} 0 \\ z \\ z \end{pmatrix} \mid z \in \mathbb{R} \right\} \\
& = & \left\{ z \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \mid z \in \mathbb{R} \right\} \ = \ \text{Vect}\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right)
\end{array}$$

f) (i) Déterminons $E_0(M_6)$. Soit $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R})$.

$$\begin{array}{rcl} X \in E_0(M_6) & \Longleftrightarrow & M_6\, X = 0\, X \\ & \Longleftrightarrow & (M_6 - 0I)\, X = 0_{\mathscr{M}_{3,1}(\mathbb{R})} \\ & \Longleftrightarrow & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ & \Longleftrightarrow & \left\{ \begin{array}{rcrcrcl} x & + & y & + & z & = & 0 \\ x & + & y & + & z & = & 0 \\ x & + & y & + & z & = & 0 \end{array} \right. \\ & \underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - L_1}{\Longleftrightarrow}} & \left\{ \begin{array}{rcrcrcl} x & + & y & + & z & = & 0 \\ & & & & 0 & = & 0 \\ & & & & 0 & = & 0 \end{array} \right. \\ & \Longleftrightarrow & \left\{ \ x = -y - z \right. \end{array}$$

On en déduit :

$$\begin{array}{rcl} E_0(M_6) & = & \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R}) \mid x = -y - z \right\} \\ & = & \left\{ \begin{pmatrix} -y - z \\ y \\ z \end{pmatrix} \mid (y, z) \in \mathbb{R}^2 \right\} \\ & = & \left\{ y \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \mid (y, z) \in \mathbb{R}^2 \right\} \\ & = & \mathrm{Vect}\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right) \end{array}$$

(ii) Déterminons $E_3(M_6)$. Soit $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R})$.

$$\begin{array}{rcl} X \in E_3(M_6) & \Longleftrightarrow & M_6\, X = 3\, X \\ & \Longleftrightarrow & (M_6 - 3I)\, X = 0_{\mathscr{M}_{3,1}(\mathbb{R})} \\ & \Longleftrightarrow & \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ & \Longleftrightarrow & \left\{ \begin{array}{rcrcrcl} -2\, x & + & y & + & z & = & 0 \\ x & - & 2\, y & + & z & = & 0 \\ x & + & y & - & 2\, z & = & 0 \end{array} \right. \\ & \underset{L_3 \leftarrow 2L_3 + L_1}{\overset{L_2 \leftarrow 2L_2 + L_1}{\Longleftrightarrow}} & \left\{ \begin{array}{rcrcrcl} -2\, x & + & y & + & z & = & 0 \\ & - & 3\, y & + & 3\, z & = & 0 \\ & & 3\, y & - & 3\, z & = & 0 \end{array} \right. \\ & \overset{L_3 \leftarrow L_3 + L_2}{\Longleftrightarrow} & \left\{ \begin{array}{rcrcrcl} -2\, x & + & y & + & z & = & 0 \\ & - & 3\, y & + & 3\, z & = & 0 \\ & & & & 0 & = & 0 \end{array} \right. \\ & \Longleftrightarrow & \left\{ \begin{array}{rcrcl} -2\, x & + & y & = & -z \\ & - & 3\, y & = & -3\, z \end{array} \right. \\ & \overset{L_1 \leftarrow 3\, L_1 + L_2}{\Longleftrightarrow} & \left\{ \begin{array}{rcrcl} -6\, x & & & = & -6\, z \\ & - & 3\, y & = & -3\, z \end{array} \right. \end{array}$$

On en déduit :

$$\begin{array}{rcl} E_3(M_6) & = & \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{M}_{3,1}(\mathbb{R}) \mid x = z \ \text{ et } \ y = z \right\} \\ & = & \left\{ \begin{pmatrix} z \\ z \\ z \end{pmatrix} \mid z \in \mathbb{R} \right\} = \left\{ z \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mid z \in \mathbb{R} \right\} = \mathrm{Vect}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) \end{array}$$

□